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SOME DEFINITE INTEGRAL INVOLVING COMPLETE ELLIPTIC INTEGRAL OF FIRST KIND IN THE FORM OF HYPERGEOMETRIC FUNCTION

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ABSTRACT
In this paper we have evaluated certain definite integral involving complete elliptic integral. The results are established in the form of Hypergeometric function.

KEY WORDS : ***Complete Elliptic Integral, Hypergeometric Function.***
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1. INTRODUCTION

The complete elliptic integral of the first kind K may thus be defined as

$$K(k)=\int_0^{\frac{\pi}{2}}\frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}=\int_0^1\frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}},(1.1)$$

or more compactly in terms of the incomplete integral of the first kind as

$$K(k)=F(\frac{\pi}{2},k)=F(1;k).(1.2)$$

It can be expressed as a power series

$$K(k)=\frac{\pi}{2}\sum_{n=0}^{\infty}[\frac{(2n)!}{2^{2n}(n!)^2}]^2k^{2n}=\frac{\pi}{2}\sum_{n=0}^{\infty}[P_{2n}(0)]^2k^{2n},(1.3)$$

where P_n is the Legendre polynomial, which is equivalent to

$$K(k)=\frac{\pi}{2}[1+(\frac{1}{2})^2k^2+(\frac{1.3}{2.4})^2k^4+.........+\{\frac{(2n-1)!!}{(2n)!!}\}^2k^{2n}+......],(1.4)$$

where $n!!$ denotes the double factorial. In terms of the Gauss hypergeometric function, the complete elliptic integral of the first kind can be expressed as

$$K(k)=\frac{\pi}{2}\,{}_2F_1(\frac{1}{2},\frac{1}{2};1;k^2)(1.5)$$

The complete elliptic integral of the first kind is sometimes called the quarter period. It can most efficiently be computed in terms of the arithmetic-geometric mean:

$$K(k)=\frac{\frac{\pi}{2}}{agm(1-k,1+k)}.(1.6)$$

A generalized hypergeometric function ${}_pF_q(a_1,...,a_p;b_1,...,b_q;z)$ is a function which can be defined in the form of a hypergeometric series, i.e., a series for which the ratio of successive terms can be written

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$$\frac{c_{k+1}}{c_k}=\frac{P(k)}{Q(k)}=\frac{(k+a_1)(k+a_2)...(k+a_p)}{(k+b_1)(k+b_2)...(k+b_q)(k+1)}z.(1.7)$$

Where $k+1$ in the denominator is present for historical reasons of notation[Koepf p.12(2.9)], and the resulting generalized hypergeometric function is written

$$ {}_pF_q\left[\begin{matrix} a_1,a_2,\cdots,a_p & ; & \\ & & z \\ b_1,b_2,\cdots,b_q & ; & \end{matrix}\right]=\sum_{k=0}^{\infty}\frac{(a_1)_k(a_2)_k\cdots(a_p)_k z^k}{(b_1)_k(b_2)_k\cdots(b_q)_k k!}(1.8)$$

where the parameters $b_1,b_2,\cdots,b_q$ are positive integers.

The ${}_pF_q$ series converges for all finite z if $p\le q$, converges for $|z|<1$ if $p=q+1$, diverges for all z , $z\ne 0$ if $p>q+1$[Luke p.156(3)].

The function ${}_2F_1(a,b;c;z)$ corresponding to $p=2,q=1$, is the first hypergeometric function to be studied (and, in general, arises the most frequently in physical problems), and so is frequently known as "the" hypergeometric equation or, more explicitly, Gauss's hypergeometric function [Gauss p.123-162]. To confuse matters even more, the term "hypergeometric function" is less commonly used to mean closed form, and "hypergeometric series" is sometimes used to mean hypergeometric function.

In mathematics, the falling factorial or Pochhammer symbol (sometimes called the descending factorial,falling sequential product, or lower factorial) is defined as the polynomial[Steffensen p.8]

$$(\varsigma)_n=\varsigma(\varsigma-1)(\varsigma-2).......(\varsigma-n+1)=\prod_{k=1}^{n}(\varsigma-k+1)=\prod_{k=0}^{n-1}(\varsigma-k)(1.9)$$

The fundamental operations of Boolean algebra are as follows:

AND (conjunction), denoted $\xi\wedge\omega$, satisfies $\xi\wedge\omega=1$ if $\xi=\omega=1$, and $\xi\wedge\omega=0$ otherwise.

OR (disjunction), denoted $\xi\vee\omega$, satisfies $\xi\vee\omega=0$ if $\xi=\omega=0$, and $\xi\vee\omega=1$ otherwise.

NOT (negation), denoted $\neg\xi$, satisfies $\neg\xi=0$ if $\xi=1$ and $\neg\xi=1$ if $\xi=0$.

2. MAIN FORMULAE OF THE INTEGRATION

$$\int_0^1\frac{x}{\sqrt{1-x^2}}K(ax)dx=\frac{1}{32}\pi[16\ {}_4F_3(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};a^2)+$$

$$+\pi a\ {}_4F_3(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};1,\frac{3}{2},2;a^2)],\text{ for } Im(a)\ne 0\vee Re(a)<0.(2.1)$$

$$\int_0^1\frac{x}{\sqrt{1-x^2}}K(\frac{x}{a})dx=\frac{1}{32a}\pi[16a\ {}_4F_3(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},\frac{3}{2};\frac{1}{a^2})+$$

$$+\pi\,{}_4F_3(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4};1,\frac{3}{2},2;\frac{1}{a^2})],\text{ for } Im(a)\ne 0\vee Re(a)<0.(2.2)$$

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$$\int_0^1 \frac{x^2}{\sqrt{1-x^4}} K(ax^2)dx = \frac{4}{225}\sqrt{2\pi}[25\,\Gamma(\frac{7}{4})^2\ {}_5F_4(\frac{1}{4},\frac{1}{4},\frac{3}{4},\frac{3}{4},\frac{3}{4};\frac{1}{2},\frac{1}{2},1,\frac{5}{4};a^2)+$$

$$+3a\,\Gamma(\frac{9}{4})^2{}_5F_4(\frac{3}{4},\frac{3}{4},\frac{5}{4},\frac{5}{4},\frac{5}{4};1,\frac{3}{2},\frac{3}{2},\frac{7}{4};a^2)], \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.3)$$

$$\int_0^1 \frac{x^4}{\sqrt{1-x^4}} K(ax^4)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{5}{4})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{5}{4};1,\frac{7}{4};a)}{8\,\Gamma(\frac{7}{4})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.4)$$

$$\int_0^1 \frac{x^5}{\sqrt{1-x^5}} K(ax^5)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{6}{5})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{6}{5};1,\frac{17}{10};a)}{10\,\Gamma(\frac{17}{10})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.5)$$

$$\int_0^1 \frac{x^7}{\sqrt{1-x^7}} K(ax^7)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{8}{7})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{8}{7};1,\frac{23}{14};a)}{14\,\Gamma(\frac{23}{14})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.6)$$

$$\int_0^1 \frac{x^9}{\sqrt{1-x^7}} K(ax^7)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{10}{7})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{10}{7};1,\frac{27}{14};a)}{14\,\Gamma(\frac{27}{14})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.7)$$

$$\int_0^1 \frac{x^{11}}{\sqrt{1-x^7}} K(ax^7)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{12}{7})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{12}{7};1,\frac{31}{14};a)}{14\,\Gamma(\frac{31}{14})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.8)$$

$$\int_0^1 \frac{x^{11}}{\sqrt{1-x^9}} K(ax^9)dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{4}{3})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{4}{3};1,\frac{11}{6};a)}{18\,\Gamma(\frac{11}{6})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.9)$$

$$\int_0^1 \frac{x^{11}}{\sqrt{1-x^{13}}} K(ax^{13})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{12}{13})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{12}{13};1,\frac{37}{26};a)}{26\,\Gamma(\frac{37}{26})}, \text{for } Im(a)\neq 0 \vee Re(a)<0.(2.10)$$

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$$\int_0^1 \frac{x^{13}}{\sqrt{1-x^{13}}} K(ax^{13})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{14}{13})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{14}{13};1,\frac{41}{26};a)}{26\,\Gamma(\frac{41}{26})}, \text{for } Im(a) \neq 0 \vee Re(a) < 0.(2.11)$$

$$\int_0^1 \frac{x^{15}}{\sqrt{1-x^{13}}} K(ax^{13})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{16}{13})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{16}{13};1,\frac{45}{26};a)}{26\,\Gamma(\frac{45}{26})}, \text{for } Im(a) \neq 0 \vee Re(a) < 0.(2.12)$$

$$\int_0^1 \frac{x^{21}}{\sqrt{1-x^{13}}} K(ax^{13})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{22}{13})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{22}{13};1,\frac{57}{26};a)}{26\,\Gamma(\frac{57}{26})}, \text{for } Im(a) \neq 0 \vee Re(a) < 0.(2.13)$$

$$\int_0^1 \frac{x^{22}}{\sqrt{1-x^{13}}} K(ax^{13})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{23}{13})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{23}{13};1,\frac{59}{26};a)}{26\,\Gamma(\frac{59}{26})}, \text{for } Im(a) \neq 0 \vee Re(a) < 0.(2.14)$$

$$\int_0^1 \frac{x^{122}}{\sqrt{1-x^{17}}} K(ax^{17})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{123}{17})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{123}{17};1,\frac{263}{34};a)}{34\,\Gamma(\frac{263}{34})}, \text{for } Im(a) \neq 0 \vee Re(a) < 0.(2.15)$$

$$\int_0^1 \frac{x^{n}}{\sqrt{1-x^{15}}} K(ax^{15})dx = \frac{\pi^{\frac{3}{2}}\Gamma(\frac{n+1}{15})\ {}_3F_2(\frac{1}{2},\frac{1}{2},\frac{n+1}{15};1,\frac{2n+17}{30};a)}{30}, \text{for } (Re(a) < 0 \vee a \notin R) \wedge Re(n) > -1.(2.16)$$

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