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(1, 2)*-QUASI η-NORMAL SPACES IN BITOPOLOGY

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ABSTRACT
In this paper, we introduce a new class of normal space called, (1, 2)-quasi η-normal space. The relationships among (1, 2)*-normal, (1, 2)*-quasi η-normal, mildly (1, 2)*-normal, (1, 2)*-quasi α-normal, (1, 2)*-mildly α-normal, (1, 2)*-α-normal and mildly (1, 2)*-η-normal spaces are investigated. Moreover, we introduce some closed functions such as (1, 2)*-πgη-closed and almost (1, 2)*-πgη-closed. Utilizing (1, 2)*-πgη-closed sets and some functions, we obtain some characterizations and preservation theorems for (1, 2)*-quasi η-normal spaces.*
KEYWORDS: *(1, 2)*-η-open, (1, 2)*-πgη-closed sets; (1, 2)*-πgη-closed, almost (1, 2)*-πgη-closed, (1, 2)*-πgη-continuous, almost (1, 2)*-πgη-continuous functions; (1, 2)*-η-normal, (1, 2)*-quasi η-normal spaces.*
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1. INTRODUCTION

The study of bitopological space was first initiated by Kelly [**7**] in 1963. By using the topological notions, namely, semi-open, α-open and pre-open sets, many new bitopological sets are defined and studied by many topologists. In 1990, Lal and Rahman [**13**] studied quasi normal spaces in topological spaces and obtained their properties. In 2000, Dontchev and Noiri [**4**] further studied quasi normal spaces in topological spaces and obtained their characterizations. In 2004, Ravi and Thivagar [**16**] studied the concept of stronger from of (1, 2)*-quotient mapping in bitopological spaces and also introduced the concepts of (1, 2)*-semi-open and (1, 2)*-α-open sets. In 2010, Arockiarani [**2**] introduced (1, 2)*-πgα-closed sets in bitopological spaces and studied some basic properties of (1, 2)*-πgα-closed sets. In 2010, K. Kayathri et al. [**6**] introduced and studied a new class of sets called regular (1, 2)*-g-closed sets and used it to obtain a new class of functions called (1, 2)*-rg-continuous and almost (1, 2)*-rg-closed functions in bitopological spaces and also obtained characterizations and preservation theorems for mildly (1, 2)*-normal spaces. In 2011, Arockiarani [**3**] introduced (1, 2)*-α-normal spaces in bitopological spaces and studied some basic properties. In 2018, H. Kumar [**8**] introduced and studied some weaker forms of quasi normal spaces in topological spaces and obtained their characterizations. In 2022, H. Kumar [**9**] introduced the concept of (1, 2)*-η-open sets and (1, 2)*-η-neighbourhood and; studied their properties. H. Kumar [**10**] introduced the concept of (1, 2)*-generalized η-closed sets and studied some basic properties of (1, 2)*-gη-closed sets. H. Kumar [**11**] introduced and studied some new functions called almost (1, 2)*-η-continuous, almost (1, 2)*-gη-continuous, almost (1, 2)*-rgη-continuous, (1, 2)*-η-closed, (1, 2)*-gη-closed, (1, 2)*-rgη-closed, almost (1, 2)*-η-closed, and almost (1, 2)*-rgη-closed functions in bitopological spaces and obtained some characterizations and preservation theorems for mildly (1, 2)*-η-normal spaces. Recently, H. Kumar [**12**] introduced and studied (1, 2)*-πgη-closed sets in bitopological spaces and obtained their properties.

2. PRELIMINARIES

Throughout the paper $(X, \Im_1, \Im_2)$, (Y, σ_1, σ_2) and $(Z, \wp_1, \wp_2)$ (or simply X, Y and Z) denote bitopological spaces.

Definition 2.1. Let S be a subset of X. Then S is said to be **$\Im_{1,2}$-open [16]** if $S = A \cup B$ where $A \in \Im_1$ and $B \in \Im_2$. The complement of a $\Im_{1,2}$-open set is **$\Im_{1,2}$-closed**.

Definition 2.2 [**16**]. Let S be a subset of X. Then
(i) the **$\Im_{1,2}$-closure** of S, denoted by $\Im_{1,2}$-cl(S), is defined as $\cap$ {F : $S \subset F$ and F is $\Im_{1,2}$-closed}; (ii) the **$\Im_{1,2}$-interior** of S, denoted by $\Im_{1,2}$-int(S), is defined as $\cup$ {F : $F \subset S$ and F is $\Im_{1,2}$-open}.

Note 2.3 [**16**]. Notice that $\Im_{1,2}$-open sets need not necessarily form a topology.

Definition 2.4. A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) regular (1, 2)*-open [16] if $A = \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}((A))$.
(ii) (1, 2)*-π-open [2] if A is the finite union of (1, 2)*-regular-open sets.
(iii) (1, 2)*-η-open [9] if $A \subset \Im_{1,2}\text{-int}(\Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int})(A)) \cup \Im_{1,2}\text{-cl}(\Im_{1,2}\text{-int}(A))$.

The complement of a regular (1, 2)*-open (resp. (1, 2)*-π-open, (1, 2)*-η-open) set is called **regular (1 , 2)*-closed** (resp. **(1, 2)*-π-closed**, **(1, 2)*-η-closed**).

The **(1, 2)*-η-closure** of a subset A of X is denoted by **(1, 2)*-η-cl(A)**, defined as the intersection of all (1, 2)*-η-closed sets containing A. The **(1, 2)*-η-interior** of S, denoted by **(1, 2)*-η-int(S)**, is defined as $\cup$ {F : F $\subset$ S and F is (1, 2)*-η-open}.

The family of all regular (1, 2)*-open (resp. regular (1, 2)*-closed, (1, 2)*-η-open, (1, 2)*-η-closed) sets in X is denoted by (1, 2)*-RO(X) (resp. (1, 2)*-RC(X), (1, 2)*-η-O(X), (1, 2)*-η-C(X)).

Remark 2.5. We have the following implications for the properties of subsets [**12**]:

regular (1, 2)*-open $\Rightarrow$ (1, 2)*-π-open $\Rightarrow$ $\Im_{1,2}$-open $\Rightarrow$ (1, 2)*-η-open

Where none of the implications is reversible.

Definition 2.6. A subset A of a bitopological space $(X, \Im_1, \Im_2)$ is called
(i) (1, 2)*-generalized η-closed (briefly **(1, 2)*-gη-closed**) [**10**] if (1, 2)*-η-cl(A) $\subset$ U whenever A $\subset$ U and U is $\Im_{1,2}$-open in X.
(ii) (1, 2)*- π generalized η-closed (briefly **(1, 2)*-πgη-closed [12]**) if (1, 2)*-η-cl(A) $\subset$ U whenever A $\subset$ U and U is (1, 2)*-π-open in X.

The complement of a (1, 2)*-gη-closed (resp. (1, 2)*-πgη-closed) set is called **(1, 2)*-gη-open** (resp. **(1, 2)*-πgη-open**).

We denote the set of all (1, 2)*-πgη-closed (resp. (1, 2)*-πgη-open) sets in $(X, \Im_1, \Im_2)$ by (1, 2)*-**πgη-C(X)** (resp. **πgη-O(X)**).
Theorem 2.7. [12]. A set A is (1, 2)*-πgη-open if and only if the following condition holds:
F $\subset$ (1, 2)*-η-int(A) whenever F is (1, 2)*-π-closed and F $\subset$ A.

3. (1, 2)*-QUASI η-NORMAL SPACES IN BITOPOLOGICAL SPACES

In this section, we introduce (1, 2)*-quasi η-normal spaces in bitopological spaces and study some basic properties of (1, 2)*-quasi η-normal spaces.

Definition 3.1. A space X is said to be **(1, 2)*-η-normal [11]** (resp. **(1, 2)*-normal, (1, 2)*-α-normal [3])** if for every pair of disjoint $\Im_{1,2}$-closed sets H and K, there exist disjoint (1, 2)*-η-open (resp. $\Im_{1,2}$-open, (1, 2)*-α-open) sets U, V of X such that H $\subset$ U and K $\subset$ V.

Definition 3.2. A space X is said to be **(1, 2)*-quasi η-normal** (resp. **(1, 2)*-quasi normal**, **(1, 2)*-quasi α-normal [2]**) if for every pair of disjoint (1, 2)*-π-closed H, K of X, there exist disjoint (1, 2)*-η-open (resp. $\Im_{1,2}$-open, (1, 2)*-α-open) sets U, V of X such that H $\subset$ U and K $\subset$ V

Definition 3.3. A space X is said to be **mildly (1, 2)*-η-normal** [**11**] (resp. **mildly (1, 2)*-normal** [**6**] **mildly (1, 2)*-α-normal**) if for every pair of disjoint H, K $\in$ (1, 2)*-RC(X), there exist disjoint (1, 2)*-η-open (resp. $\Im_{1,2}$-open, (1, 2)*-α-open) sets U, V of X such that H $\subset$ U and K $\subset$ V.

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Remark 3.4. From the definitions stated above, we obtain the following diagram.

$(1, 2)^*$-normal	$\Rightarrow$	$(1, 2)^*$-quasi normal	$\Rightarrow$	mildly $(1, 2)^*$-normal
$\Downarrow$		$\Downarrow$		$\Downarrow$
$(1, 2)^*$-α-normal	$\Rightarrow$	$(1, 2)^*$-quasi α-normal	$\Rightarrow$	mildly $(1, 2)^*$-α-normal
$\Downarrow$		$\Downarrow$		$\Downarrow$
$(1, 2)^*$-η-normal	$\Rightarrow$	$(1, 2)^*$-quasi η-normal	$\Rightarrow$	mildly $(1, 2)^*$-η-normal

Where none of the implications is reversible as can be seen from the following example:

Example 3.5. Let X = {a, b, c, d} with $\Im_1$ = {ϕ, X, {a}, {b}, {a, b}, {b, c, d}} and $\Im_2$ = {ϕ, X, {c}, {a, c, d}}. Then the pair of disjoint regular $(1, 2)^*$-closed sets H = {a} and K = {c, d}, there exist disjoint $(1, 2)^*$-η-open sets U = {a} and V = {c, d} such that $H \subset U$ and $K \subset V$. Hence (X, $\Im_1$, $\Im_2$) is mildly $(1, 2)^*$-η-normal but not mildly $(1, 2)^*$-normal, since V = {c, d} is not $\Im_{1,2}$-open set.

Example 3.6. Let X = {a, b, c} with $\Im_1$ = {ϕ, X, {a}, {a, c}} and $\Im_2$ = {ϕ, X, {c}}. Then the pair of disjoint $(1, 2)^*$-π-closed sets H = ϕ and K = {b}, there exist disjoint $(1, 2)^*$-η-open sets U = {a} and V = {b, c} such that $H \subset U$ and $K \subset V$. Hence (X, $\Im_1$, $\Im_2$) is $(1, 2)^*$-quasi η-normal but it is neither $(1, 2)^*$-quasi normal nor $(1, 2)^*$-quasi α-normal, since V = {b, c} is neither $\Im_{1,2}$-open nor $(1, 2)^*$-α-open set.

Example 3.7. Let X = {a, b, c} with $\Im_1$ = {ϕ, X, {a}} and $\Im_2$ = {ϕ, X, {b}, {a, b}}. Then the pair of disjoint $(1, 2)^*$-π-closed sets H = ϕ and K = {c}, there exist disjoint $(1, 2)^*$-η-open sets U = {b} and V = {a, c} such that $H \subset U$ and $K \subset V$. Hence (X, $\Im_1$, $\Im_2$) is $(1, 2)^*$-quasi η-normal but it is neither $(1, 2)^*$-quasi normal nor $(1, 2)^*$-quasi α-normal, since V = {b, c} is neither $\Im_{1,2}$-open nor $(1, 2)^*$-α-open set.

Theorem 3.8. For a space X, the following are equivalent:
(a) X is $(1, 2)^*$-quasi η-normal.
(b) For every pair of $(1, 2)^*$-π-open subsets U and V of X whose union is X, there exist $(1, 2)^*$-η-closed subsets G and H of X such that $G \subset U$, $H \subset V$ and $G \cup H = X$.
(c) For any $(1, 2)^*$-π-closed set A and every π-open set B in X such that $A \subset B$, there exists a $(1, 2)^*$-η-open subset U of X such that $A \subset U \subset (1, 2)^*$-$\eta$-cl(U) $\subset B$.
(d) For every pair of disjoint $(1, 2)^*$-π-closed subsets A and B of X, there exist $(1, 2)^*$-η-open subsets U and V of X such that $A \subset U$, $B \subset V$ and $(1, 2)^*$-η-cl(U) $\cap$ $(1, 2)^*$-η-cl(V) = ϕ.
Proof. (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (c), (c) $\Rightarrow$ (d) and (d) $\Rightarrow$ (a).

(a) $\Rightarrow$ (b). Let U and V be any $(1, 2)^*$-π-open subsets of a $(1, 2)^*$-quasi η-normal space X such that $U \cup V = X$. Then, X – U and X – V are disjoint $(1, 2)^*$-π-closed subsets of X. By $(1, 2)^*$-quasi η-normality of X, there exist disjoint $(1, 2)^*$-η-open subsets U_1 and V_1 of X such that $X – U \subset U_1$ and $X – V \subset V_1$. Let $G = X – U_1$ and $H = X – V_1$. Then, G and H are $(1, 2)^*$-η-closed subsets of X such that $G \subset U$, $H \subset V$ and $G \cup H = X$.

(b) $\Rightarrow$ (c). Let A be a $(1, 2)^*$-π-closed and B is a $(1, 2)^*$-π-open subsets of X such that $A \subset B$. Then, X – A and B are $(1, 2)^*$-π-open subsets of X such that $(X – A) \cup B = X$. Then, by part (b), there exist $(1, 2)^*$-η-closed sets G and H of X such that $G \subset (X – A)$, $H \subset B$ and $G \cup H = X$. Then, $A \subset (X – G)$, $(X – B) \subset (X – H)$ and $(X – G) \cap (X – H) = \phi$. Let U = X – G and V = (X – H). Then U and V are disjoint $(1, 2)^*$-η-open sets such that $A \subset U \subset X – V \subset B$. Since X – V is $(1, 2)^*$-η-closed, then we have $(1, 2)^*$-η-cl(U) $\subset (X – V)$. Thus, $A \subset U \subset (1, 2)^*$-$\eta$-cl(U) $\subset B$.

(c) $\Rightarrow$ (d). Let A and B be any disjoint $(1, 2)^*$-π-closed subset of X. Then $A \subset X – B$, where X – B is π-open. By the part (c), there exists a $(1, 2)^*$-η-open subset U of X such that $A \subset U \subset$ $(1, 2)^*$-η-cl(U) $\subset X – B$. Let V = X – $(1, 2)^*$-η-cl(U). Then, V is a $(1, 2)^*$-η-open subset of X. Thus, we obtain $A \subset U$, $B \subset V$ and $(1, 2)^*$-η-cl(U) $\cap$ $(1, 2)^*$-η-cl(V) = ϕ.

(d) $\Rightarrow$ (a). It is obvious.

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The following result is useful for giving some other characterization of $(1, 2)^*$-quasi η-normal spaces.

Theorem 3.9. For a space X, the following are equivalent:
(a) X is $(1, 2)^*$-quasi η-normal.
(b) For any disjoint $(1, 2)^*$-π-closed sets H and K, there exist disjoint $(1, 2)^*$-gη-open sets U and V such that $H \subset U$ and $K \subset V$
(c) For any disjoint $(1, 2)^*$-π-closed sets H and K, there exist disjoint $(1, 2)^*$-πgη-open sets U and V such that $H \subset U$ and $K \subset V$.
(d) For any $(1, 2)^*$-π-closed set H and any $(1, 2)^*$-π-open set V containing H, there exists a $(1, 2)^*$-gη-open set U of X such that $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset V$.
(e) For any $(1, 2)^*$-π-closed set H and any $(1, 2)^*$-π-open set V containing H, there exists a $(1, 2)^*$-πgη-open set U of X such that $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset V$.
Proof. (a) ⇒ (b), (b) ⇒ (c), (c) ⇒ (d), (d) ⇒ (e) and (e) ⇒ (a).
(a) ⇒ (b). Let X be $(1, 2)^*$-quasi η-normal space. Let H, K be disjoint $(1, 2)^*$-π-closed sets of X. By assumption, there exist disjoint $(1, 2)^*$-η-open sets U, V such that $H \subset U$ and $K \subset V$. Since every $(1, 2)^*$-η-open set is $(1, 2)^*$-gη-open, so U and V are $(1, 2)^*$-gη-open sets such that $H \subset U$ and $K \subset V$.

(b) ⇒ (c). Let H, K be two disjoint $(1, 2)^*$-π-closed sets. By assumption, there exist disjoint $(1, 2)^*$-gη-open sets U and V such that $H \subset U$ and $K \subset V$. Since $(1, 2)^*$-gη-open set is $(1, 2)^*$-πgη-open, so U and V are $(1, 2)^*$-πgη-open sets such that $H \subset U$ and $K \subset V$.

(c) ⇒ (d). Let H be any $(1, 2)^*$-π-closed set and V be any $(1, 2)^*$-π-open set containing H. By assumption, there exist disjoint $(1, 2)^*$-πgη-open sets U and W such that $H \subset U$ and $X - V \subset W$. By **Theorem 2.7**, we get $X - V \subset (1, 2)^*$-η-int(W) and $(1, 2)^*$-η-cl(U) $\cap (1, 2)^*$-η-int (W) = ϕ. Hence $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset X - (1, 2)^*$-η-int(W) $\subset V$.

(d) ⇒ (e). Let H be any $(1, 2)^*$-π-closed set and V be any $(1, 2)^*$-π-open set containing H. By assumption, there exist $(1, 2)^*$-gη-open set U of X such that $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset V$. Since, every $(1, 2)^*$-gη-open set is $(1, 2)^*$-πgη-open, there exists $(1, 2)^*$-πgη-open sets U of X such that $H \subset U \subset (1, 2)^*$-η-cl(U) $\subset V$.

(e) ⇒ (a). Let H, K be any two disjoint $(1, 2)^*$-π-closed sets of X. Then $H \subset X - K$ and $X - K$ is π-open. By assumption, there exists $(1, 2)^*$-πgη-open set G of X such that $H \subset G \subset (1, 2)^*$-η-cl(G) $\subset X - K$. Put U = $(1, 2)^*$-η-int(G), V = X – $(1, 2)^*$-η-cl(G). Then U and V are disjoint $(1, 2)^*$-η-open sets of X such that $H \subset U$ and $K \subset V$.

4. PRESERVATION THEOREMS
In this section, we shall recall the definitions of some functions used in the sequel. Further we introduce some $(1, 2)^*$-πgη-closed and almost $(1, 2)^*$-πgη-closed functions in bitopological spaces.

Definition 4.1. A function $f : X \to Y$ is said to be
(i) $(1, 2)^*$-η-continuous [**11**] if $f^{-1}(F)$ is $(1, 2)^*$-η-closed in X for every $\Im_{1,2}$-closed set F of Y;
(ii) $(1, 2)^*$-πgη-continuous [**12**] if $f^{-1}(F)$ is $(1, 2)^*$-πgη-closed in X for every $\Im_{1,2}$-closed set F of Y;
(iii) $(1, 2)^*$-π-continuous [**2**] if $f^{-1}(F)$ is $(1, 2)^*$-π-closed in X for every $\Im_{1,2}$-closed set F of Y;

Definition 4.2. A function $f : X \to Y$ is said to be
(i) almost $(1, 2)^*$-continuous [**6**] if $f^{-1}(F)$ is $\Im_{1,2}$-open in X for every $F \in (1, 2)^*$-RO(Y);
(ii) almost $(1, 2)^*$-π-continuous [**2**] if $f^{-1}(F)$ is $(1, 2)^*$-π-closed in X for every $F \in (1, 2)^*$-RC(Y);
(iii) almost $(1, 2)^*$-πgη-continuous [**12**] if $f^{-1}(F)$ is $(1, 2)^*$-πgη-closed in X for every $F \in (1, 2)^*$-RC(Y);

Definition 4.3. A function $f : X \to Y$ is said to be
(i) regular $(1, 2)^*$-closed [6] if f(F) is regular $(1, 2)^*$-closed in Y for every $\Im_{1,2}$-closed set F of X;
(ii) $(1, 2)^*$-η-closed [**11**] if f(F) is $(1, 2)^*$-η-closed in Y for every $\Im_{1,2}$-closed set F of X;
(iii) $(1, 2)^*$-gη-closed [**11**] if f(F) is $(1, 2)^*$-gη-closed in Y for every $\Im_{1,2}$-closed set F of X;
(iv) $(1, 2)^*$-πgη-closed if f(F) is $(1, 2)^*$-πgη-closed in Y for every $\Im_{1,2}$-closed set F of X.

Definition 4.4. A function $f : X \to Y$ is said to be
(i) $(1, 2)^*$-rc-preserving [**6**] if f(F) is regular $(1, 2)^*$-closed in Y for every $F \in (1, 2)^*$-RC(X);
(ii) almost $(1, 2)^*$-closed [**11**] if f(F) is $\psi_{1,2}$-closed in Y for every $F \in (1, 2)^*$-RC(X);

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(iii) almost (1, 2)*- η-closed [**11**] if f(F) is (1, 2)*-η-closed in Y for every F ∈ (1, 2)*-RC(X);
(iv) almost (1, 2)*-gη-closed [**11**] if f(F) is (1, 2)*-gη-closed in Y for every F ∈ (1, 2)*-RC(X);
(v) almost (1, 2)*-πgη-closed if f(F) is (1, 2)*-πgη-closed in Y for every F ∈ (1, 2)*-RC(X);

Remark 4.5. From the definitions stated above, we obtain the following diagram.

regular (1, 2)*-closed	⇒	(1, 2)*-rc-preserving
⇓		⇓
(1, 2)*-closed	⇒	almost (1, 2)*-closed
⇓		⇓
(1, 2)*-η-closed	⇒	almost (1, 2)*-η-closed
⇓		⇓
(1, 2)*-gη-closed	⇒	almost (1, 2)*-gη-closed
⇓		⇓
(1, 2)*-πgη-closed	⇒	almost (1, 2)*-πgη-closed

The following examples enable us to realize that none of the implications in the above diagram is reversible.

Example 4.6. Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, ψ_1 = {ϕ, Y, {a, b}} and ψ_2 = {ϕ, Y, {a}}. Define f : X → Y as f(a) = b; f(b) = a; f(c) = c. Clearly f is (1, 2)*-gη-closed as well as almost (1, 2)*-gη-closed. It is also almost (1, 2)*-πgη-closed. But it is neither (1, 2)*-closed nor almost (1, 2)*-closed. It is neither (1, 2)*-η-closed nor almost (1, 2)*-η-closed.

Example 4.7. Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, ψ_1 = {ϕ, Y, {a, b}} and ψ_2 = {ϕ, Y, {a}}. Define f : X → Y as f(a) = b, f(b) = c, f(c) = a. But it is neither (1, 2)*-η-closed nor almost (1, 2)*-η-closed. It is neither (1, 2)*-gη-closed nor almost (1, 2)*-gη-closed.

Example 4.8. Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {a, c}}, ψ_1 = {ϕ, Y, {a, b}} and ψ_2 = {ϕ, Y, {a}}. Define f : X → Y as f(a) = b; f(b) = a; f(c) = c. Clearly f is almost (1, 2)*-closed as well as almost (1, 2)*-gη-closed, but it is not (1, 2)*-closed. It is also almost (1, 2)*-πgη-closed

Example 4.9. Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, ψ_1 = {ϕ, Y, {b}, {c}, {b, c}} and ψ_2 = {ϕ, Y, {a, b}}. Define f : X → Y as f(a) = c, f(b) = b, f(c) = a. Clearly f is (1, 2)*-closed as well as almost (1, 2)*-closed. It is (1, 2)*-η-closed as well as almost (1, 2)*-gη-closed. But it is neither regular (1, 2)*-closed nor (1, 2)*-rc-preserving.

Example 4.10. Let X = Y = {a, b, c}, $\Im_1$ = {ϕ, X, {a}}, $\Im_2$ = {ϕ, X, {b}}, ψ_1 = {ϕ, Y, {a}, {a, c}} and ψ_2 = {ϕ, Y, {c}}. Define f : X → Y as f(a) = a; f(b) = c; f(c) = b. Clearly f is (1, 2)*-rc-preserving as well as almost (1, 2)*-η-closed.

Theorem 4.11. If f : X → Y is an almost (1, 2)*-π-continuous and (1, 2)*-πgη-closed function, then f(A) is (1, 2)*-πgη-closed in Y for every (1, 2)*-πgη-closed set A of X.
Proof. Let A be any (1, 2)*-πgη-closed set of X and V be any (1, 2)*-π-open set of Y containing f(A). Since f is almost (1, 2)*-π-continuous, f^{-1}(V) is (1, 2)*-π-open in X and A ⊂ f^{-1}(V). Therefore, we have (1, 2)*-η-cl(A) ⊂ f^{-1}(V) and hence f((1, 2)*-η-cl(A)) ⊂ V. Since f is (1, 2)*-πgη-closed, f((1, 2)*-η-cl(A)) is (1, 2)*-πgη-closed in Y and hence we obtain (1, 2)*-η-cl(f(A)) ⊂ (1, 2)*-η-cl(f((1, 2)*-η-cl(A))) ⊂ V. Hence f(A) is (1, 2)*-πgη-closed in Y.

Theorem 4.12. A surjection f : X → Y is almost (1, 2)*-πgη-closed if and only if for each subset S of Y and each U ∈ (1, 2)*-RO(X) containing f^{-1}(S), there exists a (1, 2)*-πgη-open set V of Y such that S ⊂ V and f^{-1}(V) ⊂ U.
Proof. Necessity. Suppose that f is almost (1, 2)*-πgη-closed. Let S be a subset of Y and U ∈ (1, 2)*-RO(X) containing f^{-1}(S). If V = Y – f (X – U), then V is a (1, 2)*-πgη-open set of Y such that S ⊂ V and f^{-1}(V) ⊂ U.

Sufficiency. Let F be any regular (1, 2)*-closed set of X. Then f^{-1}(Y – f(F)) ⊂ (X – F) and (X – F) ∈ (1, 2)*-RO(X). There exists a (1, 2)*-πgη-open set V of Y such that Y – f(F) ⊂ V and f^{-1}(V) ⊂ (X – F). Therefore, we have f(F) ⊃ (Y – V) and F ⊂ X – f^{-1}(V) ⊂ f^{-1} (Y – V) . Hence we obtain f(F) = Y – V and f(F) is (1, 2)*-πgη-closed in Y, which shows that f is almost (1, 2)*-πgη-closed.

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Theorem 4.13. If f : X → Y is an almost $(1, 2)^*$-πgη-continuous, $(1, 2)^*$-rc-preserving injection and Y is $(1, 2)^*$-quasi η-normal then X is $(1, 2)^*$-quasi η-normal.
Proof. Let A and B be any disjoint $(1, 2)^*$-π-closed sets of X. Since f is a $(1, 2)^*$-rc-preserving injection, f(A) and f(B) are disjoint $(1, 2)^*$-π-closed sets of Y. Since Y is $(1, 2)^*$-quasi η-normal, there exist disjoint $(1, 2)^*$-η-open sets U and V of Y such that f(A) ⊂ U and f(B) ⊂ V.

Now if G = $(1, 2)^*$-int($(1, 2)^*$-cl(U)) and H = $(1, 2)^*$-int($(1, 2)^*$-cl(V)). Then G and H are regular $(1, 2)^*$-open sets such that f(A) ⊂ G and f(B) ⊂ H. Since f is almost $(1, 2)^*$-πgη-continuous, f^{-1}(G) and f^{-1}(H) are disjoint $(1, 2)^*$-πgη-open sets containing A and B which shows that X is $(1, 2)^*$-quasi η-normal.

Theorem 4.14. If f : X → Y is $(1, 2)^*$-π-continuous, almost $(1, 2)^*$-η-closed surjection and X is $(1, 2)^*$-quasi η-normal space then Y is $(1, 2)^*$-η-normal.
Proof. Let A and B be any two disjoint closed sets of Y. Then f^{-1}(A) and f^{-1}(B) are disjoint $(1, 2)^*$-π-closed sets of X. Since X is quasi $(1, 2)^*$-η-normal, there exist disjoint $(1, 2)^*$-η-open sets U and V such that f^{-1}(A) ⊂ U and f^{-1}(B) ⊂ V.

Let G = $(1, 2)^*$-int($(1, 2)^*$-cl(U)) and H = $(1, 2)^*$-int($(1, 2)^*$-cl(V)). Then G and H are disjoint regular $(1, 2)^*$-open sets of X such that f^{-1}(A) ⊂ G and f^{-1}(B) ⊂ H. Now, we set K = Y – f(X – G) and L = Y – f (X – H). Then K and L are $(1, 2)^*$-η-open sets of Y such that A ⊂ K, B ⊂ L, f^{-1}(K) ⊂ G and f^{-1}(L) ⊂ H. Since G and H are disjoint, K and L are disjoint. Since K and L are $(1, 2)^*$-η-open and we obtain A ⊂ $(1, 2)^*$-η-int(K), B ⊂ $(1, 2)^*$-η-int(L) and $(1, 2)^*$-η-int(K) ∩ $(1, 2)^*$-η-int(L) = ϕ. Therefore, Y is $(1, 2)^*$-η-normal.

Theorem 4.15. Let f : X → Y be an almost $(1, 2)^*$-π-continuous and almost $(1, 2)^*$-πgη-closed surjection. If X is $(1, 2)^*$-quasi η-normal space then Y is $(1, 2)^*$-quasi η-normal.
Proof. Let A and B be any disjoint $(1, 2)^*$-π-closed sets of Y. Since f is almost $(1, 2)^*$-π-continuous, f^{-1}(A) and f^{-1}(B) are disjoint $(1, 2)^*$-π-closed sets of X. Since X is $(1, 2)^*$-quasi η-normal, there exist disjoint $(1, 2)^*$-η-open sets U and V of X such that f^{-1}(A) ⊂ U and f^{-1}(B) ⊂ V.

Put G = $(1, 2)^*$-int($(1, 2)^*$-cl(U)) and H = $(1, 2)^*$-int($(1, 2)^*$-cl(V)). Then G and H are disjoint regular $(1, 2)^*$-open sets of X such that f^{-1}(A) ⊂ G and f^{-1}(B) ⊂ H. By **Theorem 4.12,** there exist $(1, 2)^*$-gη-open sets K and L of Y such that A ⊂ K, B ⊂ L, f^{-1}(K) ⊂ G and f^{-1}(L) ⊂ H. Since G and H are disjoint, so are K and L by **Theorem 2.7,** A ⊂ $(1, 2)^*$-η-int(K), B ⊂ $(1, 2)^*$-η-int(L) and $(1, 2)^*$-η-int(K) ∩ $(1, 2)^*$-η-int(L) = ϕ. Therefore, Y is $(1, 2)^*$-quasi η-normal.

Corollary 4.16. If f : X → Y is an almost $(1, 2)^*$-continuous and almost $(1, 2)^*$-closed surjection and X is a $(1, 2)^*$-normal space, then Y is $(1, 2)^*$-quasi η-normal.
Proof. Since every almost $(1, 2)^*$-closed function is almost $(1, 2)^*$-πgη-closed by **Theorem 4.15,** Y is $(1, 2)^*$-quasi η-normal.

5. CONCLUSION

In this paper, we introduce a new class of normal space called, $(1, 2)^*$-quasi η-normal space. The relationships among $(1, 2)^*$-normal, $(1, 2)^*$-quasi η-normal, mildly $(1, 2)^*$-normal, $(1, 2)^*$-quasi α-normal, $(1, 2)^*$-mildly α-normal, $(1, 2)^*$-α-normal and mildly $(1, 2)^*$-η-normal spaces are investigated. Moreover, we introduce some functions such as $(1, 2)^*$-η-closed, $(1, 2)^*$-gη-closed, $(1, 2)^*$-πgη-closed, almost $(1, 2)^*$-η-closed, almost $(1, 2)^*$-gη-closed and almost $(1, 2)^*$-πgη-closed. Utilizing $(1, 2)^*$-πgη-closed sets and some functions, we obtain some characterizations and preservation theorems for $(1, 2)^*$-quasi η-normal spaces. This idea can be extended to ordered topological, ordered bitopological and fuzzy topological spaces etc.

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