SJIF Impact Factor (2023): 8.574| ISI I.F. Value: 1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 8 | Issue: 10 | October 2023 - Peer Reviewed Journal

AN OVERVIEW OF SUMUDU TRANSFORM WITH ORDINARY DIFFERENTIAL EQUATIONS

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Article DOI: **https://doi.org/10.36713/epra14623**
DOI No: 10.36713/epra14623

ABSTRACT
This paper introduces a new integral transform called the Sumudu transformation, which offers unique advantages. Unlike other transforms, Sumudu allows for the preservation of units, making it highly relevant in engineering applications. This paper defines the Sumudu transform, explores its properties, and demonstrates its applicability in solving linear ordinary differential equations. The Sumudu transform's ability to handle non-constant coefficients in equations is highlighted, making it a valuable tool for engineers and scientists dealing with a wide range of differential equations. Ultimately, Sumudu transformation proves to be a versatile and powerful technique for solving differential equations in both the time and frequency domains.
KEYWORDS: *Sumudu Transforms, ODEs, Shift Theorems, Differential Equations*

1 INTRODUCTION

There are numerous integral transforms, such as Laplace, Fourier, Mellin, and Hankel, to name a few, to solve differential equations and control engineering problems. Of these, the Laplace transformation is the most widely used. The essence of the Laplace transformation is the mapping of a t-domain function such as f(t) to a Laplace domain (s-domain) function F(s) by an integral transformation. By this, the differentiation and integration operations in the t-domain are made equivalent to multiplication and division by s in the s-domain. The variable s and the transformed function F(s) in the s-domain are treated as dummies in this process, and their physical significance is not questioned. A new integral transformation (which is termed the Sumudu transformation) is introduced in this paper. Watagula's [1993] was probably the first paper to introduce the Sumudu Transform and motivate its use. Its simple formulation and direct applications to ordinary differential equations immediately sparked interest in this new tool. In Sumudu transform, the differentiation and integration operations in the t-domain are made equivalent to division and multiplication by u in a u-domain. This makes it possible to treat the variable u and transformed function F(u) as replicas of t and f(t), respectively. It is even possible to express them in the same engineering units as t and f(t) so that the consistency of units in a differential equation describing a physical process can be maintained even after the transformation.

It is known that the Sumudu transform has many other interesting properties. Some of which are:
1. The unit-step function in the t-domain is transformed to unity in the u-domain.
2. Scaling of the function f(t) in the t-domain is equivalent to scaling of F(u) in the u-domain by the same scale factor.
3. The limit of f(t) as t tends to zero is equal to the limit of F(u) as u tends to zero.
4. For several cases, the limit of F(t) as t tends to infinity is the same as the limit of F(u) as u tends to infinity.
5. The slope of the function f(t) at t = 0 is the same as the slope of F(u) at u = 0. [8]

Our purpose in this study is to show the applicability of this interesting new transform and its efficiency in solving the linear ordinary differential equations. The first part of this paper gives the definition of the Sumudu transform. The properties resulting from the definition are given in the next section, illustrating its simplicity. Application of Sumudu Transform to the differential equation as worked examples are provided in the last section.

2 Definition of Sumudu Transform

The Sumudu Transform is defined by the formula

$$F(u)+S[f(t)\,;\,u]=\frac{1}{u}\int_{0}^{\infty}e^{\frac{-t}{u}}f(t)du \quad u\in\left(\tau_1,\tau_2\right)$$

Over the set of functions, $A=\{f(t)|\ \exists M_{,\tau 1,\tau 2}>0,\ |f(t)|<Me^{\frac{t}{\tau_j}},\ if\ t\in(-1)^j\times(0,\infty]\}$

For a given function in the set A, the constant M must be finite, while τ_1 and τ_2 need not exist simultaneously, and each may be infinite. [5]

3 Sumudu Transform of the nth Derivative

Theorem 3.1. Let *f(t)* be in A, and let $G^{(n)}(u)$ denote the Sumudu transform of the *n*th derivative, $f^{(n)}(t)$ of *f(t)*, then for n ≥ 1 The *Sumudu transform* is defined by the following formula:

$$G^n(u) = \frac{G(u)}{u^n} - \sum_{k=0}^{n-1} \frac{f^k(0)}{u^{n-k}}$$

Proof. By Induction.

Base Case for n = 1 :

$$G' = \frac{G(u)}{u'} - \frac{f(0)}{u'}$$

Since by definition:

$$G'(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt$$

We can replace the G'(u) in the equation above:

$$\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt = \frac{\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)dt}{u} - \frac{f(0)}{u}$$

Doing Integration by parts:

$$= \frac{1}{u^2}\left[\left[f(t)\frac{e^{\frac{-t}{u}}}{-\frac{1}{u}}\right]_0^\infty - \int_0^\infty f'(t)\frac{e^{\frac{-t}{u}}}{-\frac{1}{u}}dt\right] - \frac{f(0)}{u}$$

$$= \frac{1}{u^2}\left[\left[-u\,f(t)e^{\frac{-t}{u}}\right]_0^\infty + u\int_0^\infty e^{\frac{-t}{u}} f'(t)dt\right] - \frac{f(0)}{u}$$

$$= -\frac{1}{u}\left[f(t)e^{\frac{-t}{u}}\right]_0^\infty + \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt - \frac{f(0)}{u}$$

$$= -\frac{f(0)}{u} + \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt - \frac{f(0)}{u}$$

$$\frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt - \frac{f(0)}{u} = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f'(t)dt - \frac{f(0)}{u}$$

Induction Step.

Assuming it's true for *n*, we'll now prove that it's also true for *n+1*.

Proof.

$$G^{n+1}(u) = \frac{G(u)}{u^{n+1}} - \sum_{k=0}^{n-1} \frac{f^{n+1}(0)}{u^{n+1-k}}$$

$$G^{n+1}(u) = S\left[f^{n+1}(t)\right]$$

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$$G^{n+1}(u) = S\left[(f^{n}(t))'\right]$$

$$G^{n+1}(u) = \frac{S\left[f^{n}(t)\right] - f^{n}(0)}{u}$$

$$G^{n+1}(u) = \frac{G^{n}(u) - f^{n}(0)}{u}$$

$$G^{n+1}(u) = \frac{\frac{G(u)}{u^{n}} - \sum_{k=0}^{n-1} \frac{f^{k}(0)}{u^{n-k}} - f^{n}(0)}{u}$$

$$G^{n+1}(u) = \frac{G(u)}{u^{n+1}} - \sum_{k=0}^{n} \frac{f^{k}(0)}{u^{n+1-k}}$$

$$G^{n+1}(u) - \sum_{k=0}^{n} \frac{f^{k}(0)}{u^{n+1-k}} = \frac{G(u)}{u^{n+1}} - \sum_{k=0}^{n} \frac{f^{k}(0)}{u^{n+1-k}}$$

4 Existence of Sumudu Transforms

Theorem 4.1. If f is of exponential order such that $|f(t)| < Me^{\frac{t}{\tau_j}}$, where $\exists M_{\tau 1, \tau 2} > 0$ and $t \in (-1)^{j} \times (0, \infty]\}$ then the Sumudu Transform $S[f(t)] = F(u)$ exists. The defining integral for F exists at points $\frac{1}{u} = \frac{1}{n} + \frac{i}{\tau}$ in the right half plane $\eta >$ K and $\varsigma >$ L. [2]

Proof. Since

$$e^{\alpha + i\beta} = e^{t}(cos\alpha + isin\beta)$$

we can use $\frac{1}{u} = \frac{1}{n} + \frac{i}{\tau}$ and can express $F(u) = \int_{0}^{\infty} e^{\frac{-t}{u}} f'(t)dt$ in terms of sine and cosine functions.

$$\int_{0}^{\infty} f(t)e^{-(\frac{1}{\eta} + i\frac{1}{\tau})t} dt$$

$$\int_{0}^{\infty} f(t)e^{-(\frac{t}{\eta})}(cos\frac{t}{\tau} + isin\frac{t}{\tau})dt$$

$$\int_{0}^{\infty} f(t)e^{-(\frac{t}{\eta})}(cos\frac{t}{\tau})dt + i\int_{0}^{\infty} f(t)e^{-(\frac{t}{\eta})}(-\ sin\frac{t}{\tau})dt$$

$$\int_{0}^{\infty} f(t)e^{-(\frac{t}{\eta})}(cos\frac{t}{\tau})dt - i\int_{0}^{\infty} f(t)e^{-(\frac{t}{\eta})}(sin\frac{t}{\tau})dt$$

Now since

$$A = \{f(t)|\ \exists M_{,\tau 1, \tau 2} > 0,\ |f(t)| < Me^{\frac{t}{\tau_j}},\ if\ t \in (-1)^{j} \times (0,\infty]\}$$

Then for all values of $\frac{1}{n} + \frac{i}{\tau}$ we have

$$|f(t)| < Me^{\frac{-t}{\tau}} \text{ if } t \leq 0$$

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$$|f(t)| < Me^{\frac{t}{\tau}} \text{ if } t > 0$$

$$\int_0^\infty |f(t)|\,|cos\frac{t}{\tau}|e^{-(\frac{t}{\eta})}dt \le \int_0^\infty Me^{\frac{t}{\tau}} \cdot e^{-(\frac{t}{\eta})}dt$$

$$\int_0^\infty |f(t)|\,|cos\frac{t}{\tau}|e^{-(\frac{t}{\eta})}dt \le \int_0^\infty Me^{(\frac{1}{\tau}-\frac{1}{\eta})t}dt$$

$$\int_0^\infty |f(t)|\,|cos\frac{t}{\tau}|e^{-(\frac{t}{\eta})}dt \le M\int_0^\infty e^{(\frac{\eta-\tau}{\eta\tau})t}dt$$

$$\int_0^\infty |f(t)|\,|cos\frac{t}{\tau}|e^{-(\frac{t}{\eta})}dt \le M\frac{\eta\tau}{\eta-\tau}\left[e^{(\frac{\eta-\tau}{\eta\tau})t}\right]_0^\infty$$

$$\int_0^\infty |f(t)|\,|cos\frac{t}{\tau}|e^{-(\frac{t}{\eta})}dt \le (\frac{M\eta\tau}{\eta-\tau}) \text{ for } \frac{1}{\eta} > \frac{1}{\tau}$$

which imply that the integrals defining the real and imaginary parts of F exist for value of Re$(\frac{1}{u}) > (\frac{1}{\tau})$, completing the proof.

5 Amplification of the Coefficients of the Power Series Function

Lemma 1. We will be using the *Gamma Function* and its properties in the following theorem:

$$\gamma(n) = \int_0^\infty u^{n-1}e^{-t}dt \quad n > 0$$

$$\gamma(n+1) = \int_0^\infty u^{n}e^{-t}dt \quad n > 0$$

Also,

$$\gamma(n+1) = n!$$

Theorem 5.1. The *Sumudu Transform* amplifies the coefficients of the *Taylor Power series function,*

$$f(t) = \sum_{n=0}^{\infty} a_n t^n$$

by sending it to the power series function,

$$G(u) = \sum_{n=0}^{\infty} n!\, a_n t^n$$

Proof. Since the function f(t) exists and is analytic, its Sumudu S{f(t)] exists, and we can deduce that its Taylor polynomial also exists,

If $f(t) = \sum_{n=0}^{\infty} a_n t^n$ in some interval I ⊂ R, then by Taylor functions expansion theorem, since

$$f(t) = \sum_{n=0}^{\infty} \frac{f^n(0)}{n!} t^n$$

and transforming our definition of Sumudu to a different form:

$$S[f(t);\ u] = \frac{1}{u}\int_0^{\infty} e^{\frac{-t}{u}} f(t)dt$$

let t' = $\frac{t}{u}$ and $t = t'\,u$. Replacing t in the equation:

$$= \frac{u}{u}\int_0^{\infty} e^{\frac{-t''u}{u}} f(t'\,u)\,dt'$$

$$= \int_0^{\infty} e^{-t'} f(t'\,u)dt'$$

$$= \int_0^{\infty} f(tu)e^{-t}dt$$

$$G(u) = S[f(t)] = \int_0^{\infty} f(ut)e^{-t}dt \quad u \in (\tau_1, \tau_2)$$

we have,

$$S[f(t)] = \int_0^{\infty} \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(ut)^n e^{-t}dt$$

We can factor out the infinite series from the improper integral because the Taylor polynomial converges absolutely when the function f(t) in analytic,

$$= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(u)^n \int_0^{\infty} t^n e^{-t}dt$$

$$= \sum_{n=0}^{\infty} \frac{f^n(0)}{n!}(u)^n \gamma(n + 1)$$

$$= \sum_{n=0}^{\infty} f^n(0)u^n$$

And a nice consequence of this is,

$$S\left[(1 + t)^m\right] = S(\sum_{n=0}^{\infty} C_n^m t^n)$$

$$= S(\sum_{n=0}^{m} \frac{m!}{n!(m-n)!} u^n)$$

$$= S(\sum_{n=0}^{m} \frac{m!}{(m-n)!} u^n)$$

$$= S(\sum_{n=0}^{m} P_n^m t^n)$$

The Sumudu Transform sends combinations, C_n^m, into permutations, P_n^m, and hence may seem to incur more order into discrete systems.

6 Inverse Sumudu Transform of Power Series

Theorem 6.1. The inverse discrete Sumudu transform, *f(t),* of the power series $G(u) = \sum_{n=0}^{\infty} b_n u^n$ is given by:

$$S^{-1}[G(u)] = f(t) = \sum_{n=0}^{\infty} \frac{1}{n!} b_n t^n$$

Proof. We say that if the Sumudu transform is applied to the power series $f(t) = \sum_{n=0}^{\infty} a_n t^n$, we get

$$G(u) = S[f(t)] = \sum_{n=0}^{\infty} n!\, a_n u^n$$

Now as we can see from theorem 5.1 that the Sumudu Transform amplifies the coefficients of the power series. Thus by intuition, the inverse of Sumudu should do the opposite. At least for the discrete case uptil null functions. This is a direct implication of theorem 5.1 for discrete case.

$$S^{-1}[S[f(t)]] = f(t)$$

where f(t) is a proper series function; hence Sumudu Transform is simply

$$S^{-1}[G(u)] = \sum_{n=0}^{\infty} \frac{1}{n!} b_n t^n = f(t)$$

7 Sumudu Multiple Shift Theorems

Theorem 7.1. If *G(u)* denotes the Sumudu Transform of *f(t)* in A, then the Sumudu Transform of the function *t^n f(t)* is given by

$$S\left[t^n f(t)\right] = u^n \sum_{k=0}^{n} a_k^n u^k G_k(u)$$

where $a_0^n = n!,\ a_n^n = 1,\ a_1^n = n!\, n,\ a_n - 1^n = n^2$, and for k=2, 3, ……, n

Proof. Proof by Induction

Base Case for n=0

$$S\left[t^0 f(t)\right] = u^0 \sum_{k=0}^{0} a_k^0 u^k G_k(u)$$

$$S\left[t^0 f(t)\right] = a_0^0 u^0 G_k(u)$$

$$S[f(t)] = G(u)$$

hence the statement holds for n=0. To perform the induction step, we assume that it's true for n, and show that it carries to n+1. Setting $W(u) = S\left[t^n f(t)\right]$, we have

$$S\left[t^n f(t)\right] = S\left[t\left[t^n f(t)\right]\right] = u\frac{d(uW(u))}{du} = uW(u) + u^2\frac{d(W(u))}{du}$$

$$uW(u) + u^2\frac{d(W(u))}{du} = u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^2\frac{d}{du}\sum_{k=0}^{n} a_k^n u^k G_k(u)$$

$$= u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^2\sum_{k=0}^{n} a_k^n[(n+k)]u^{n+k+1} G_k(u)$$

$$= u^{n+1}\sum_{k=0}^{n} a_k^n u^k G_k(u) + u^{n+1}\sum_{k=0}^{n} a_k^n[n+k]u^k G_k(u) + u^{k+1} G_{k+1}(u)$$

$$= u^{n+1}\sum_{k=0}^{n} (n+k+1)a_k^n u^k G_k(u) + = u^{n+1}\sum_{k=0}^{n} a_k^n u^{k+1} G_k(u)$$

Noting that for k < 0, or k > n, $a_k^n = 0$, we can rewrite the previous equation as:

$$= u^{n+1}\sum_{k=0}^{n+1} (n+k+1)a_k^n u^k G_k(u) + = u^{n+1}\sum_{k=0}^{n+1} a_{k-1}^n u^k G_k(u)$$

$$= u^{n+1}\sum_{k=0}^{n+1}\left[(n+k+1)a_k^n + a_{k-1}^n\right]u^k G_k(u)$$

And since,

$$a_k^n = a_{k-1}^{n-1} + (n+k)a_k^{n-1}$$

$$= u^{n+1}\sum_{k=0}^{n+1} a_k^{n+1} u^k G_k(u)$$

Theorem 7.2. Let *G(u)* denote the Sumudu Transform of the function *f(t)* in A. Let $f^n(t)$ denote the nth derivative of *f(t)* with respect to t. And let $G_n(u)$ denote the nth derivative of *G(u)* with respect to u. Then the Sumudu Transform of the function $t^n f^n(t)$ is given by:

$$S\left[t^n f^{(n)}(t)\right] = u^n G_n(u)$$

Proof. Being the Sumudu Transform of *f(t)*,

$$G(u) = \frac{1}{u}\int_0^\infty e^{\frac{-t}{u}} f(t)dt$$

let $t' = \frac{t}{u}$ and $t = t'u$. Replacing t in the equation:

$$= \frac{u}{u}\int_0^\infty e^{\frac{-t'' u}{u}} f(t' u)\, dt'$$

$$= \int_0^\infty e^{-t'} f(t' u) dt'$$

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$$= \int_0^\infty f(tu)e^{-t}dt$$

$$G(u) = S[f(t)] = \int_0^\infty f(ut)e^{-t}dt \quad u \in (\tau_1, \tau_2)$$

Therefore, for n = 1, 2, 3, …., we have

$$G(u) = \int_0^\infty \frac{d^n}{du^n} f(ut)e^{-t}dt = \int_0^\infty t^n f^{(n)}(ut)e^{-t}dt$$

$$\frac{1}{u^n}\int_0^\infty (ut)^n f^{(n)}(ut)e^{-t}dt = \frac{1}{u^n} S\left[t^n f^{(n)}(t)\right]$$

Upon multiplying both sides of the previous equation by u^n, we obtain,

$$S\left[t^n f^{(n)}(t)\right] = u^n G_n(u)$$

Sumudu Transformation applies to ordinary linear differential equations with constant terms, the same way as Laplace Transformation does but in case of non-constant coefficients present in the equation, the Laplace Transform does not apply and we have to resort to using Sumudu Transform instead. In our findings below, we will try to build up a global Sumudu shift theorem about $S[t^n f^n(t)]$. Now, using Theorem 7.1 and 7.2, we can generate the following results.

Lemma 2. Let $G_n(u)$ denote the n^th derivative of $G_n(u) = S[t^n f^n(t)]$, then,

Case 1: where n = 2

$$S\left[t^2 f''\right] = u^2\left[2G_1(u) + uG_2(u)\right]$$

Proof.

$$S\left[t^2 f''\right] = S[t * (tf)']$$

$$= u^2\left[a_0^1 u^0 S[tf'] + a_1^1 u^1 [f' + tf'']\right]$$

$$= u^2\left[a_0^1 u^0 G_1(u) + a_1^1 u^1 [S[f'] + S[tf'']]\right]$$

$$= u^2\left[a_0^1 u^0 G_1(u) + a_1^1 u^1 G_1(u) + a_1^1 u^1 S[tf'']\right]$$

From theorem 7.2 we know that

$$S\left[t^2 f''\right] = u^2 \cdot G_2(u)$$

Here G(u) denotes the Sumudu Transform of the function f(t) and $G_n(u)$ denotes the n^th derivative of G(u) with respect to function f(t).

Then,

$$S\left[t^2 f''\right] = u^2\left[(a_0^1 u^1 + a_1^1 u^1)G_1(u) + a_1^1 u^2 G_2(u)\right]$$

where $a_0^n = n!$, $a_n^n = 1$, $a_1^n = n!\,n$, $a_n - 1^n = n^2$, and for k = 2, 3, ….., n

$$a_k^n = a_{k-1}^{n-1} + (n + k)a_k^{n-1}$$

and finally we get,

$$S\left[t^2 f'\right] = u^2\left[2G_1(u) + uG_2(u)\right]$$

Case 2: where n = 3

$$S\left[t^3 f'\right] = u^3\left[6G_1(u) + 6uG_2(u) + u^2G_3(u)\right]$$

Proof.

$$S\left[t^3 f'''\right] = S\left[t^2[tf'']\right]$$

$$= u^2\left[a_0^2u^0S[tf'] + a_1^2u^1S[f' + tf''] + a_2^2u^2S[f' + f'' + tf''']\right]$$

$$= u^2\left[2u^1G_1(u) + 4u^1S[f'] + S[tf''] + u^2S[f'] + u^2S[f''] + u^2S[tf''']\right]$$

$$= u^2\left[2u^1G_1(u) + 4u^1G_1(u) + 4u^2G_2(u) + u^2G_2(u) + u^2G_2(u) + u^3G_3(u)\right]$$

$$= u^2\left[6u^1G_1(u) + 6u^2G_2(u) + u^3G_3(u)\right]$$

Factorizing u from the inner equation:

$$= u^3\left[6G_1(u) + 6uG_2(u) + u^2G_3(u)\right]$$

and finally we get,

$$S\left[t^3 f'''\right] = u^3\left[6G_1(u) + 6uG_2(u) + u^2G_3(u)\right]$$

Case 3: where n = 4

$$S\left[t^4 f''''\right] = u^4\left[12G_2(u) + 8G_3(u) + u^2G_4(u)\right]$$

Proof.

According to Theorem 7.1,

$$S\left[t^2\left[t^2 f''''\right]\right] = u^2\left[a_0^2u^0S(t^2f'') + a_1^2u^1S\left[(t^2f''')'\right] + a_2^2u^2S\left[(t^2f'')''\right]\right]$$

$$= u^2\left[a_0^2u^0S(t^2f'') + a_1^2u^1S(2tf'' + t^2f''') + a_2^2u^2S(2tf''' + 2f'' + 2tf''' + t^2f''''\right]$$

$$= u^2\left[1u^2G_2(2) + 4uS\left[2tf'' + t^2f'''\right] + u^2S\left[2tf''' + 2f'' + 2tf''' + t^2f''''\right]\right]$$

$$= u^2\left[u^2G_2(u) + 4u\left[2uG_2(u) + u^2G_3(u)\right] + u^2\left[2uG_3(u) + 2G_2(u) + 2uG_3(u) + u^2G_4(u)\right]\right]$$

$$= u^2\left[u^2G_2(u) + 8u^2G_2(u) + 4u^3G_3(u)\right] + 2u^3G_3(u) + 2u^2G_2(u) + 2u^3G_3(u) + u^4G_4$$

$$= u^2\left[12u^2G_2(u) + 8u^3(u)\right] + u^4G_4(u)$$

$$= u^4\left[12G_2(u) + 8uG_3(u) + u^2G_4(u)\right]$$

Theorem 7.3. Following the results of the previous sections, we can build up a general thearon for the Sumudu Transform of a function, $t^nf^m(t)$, given by

$$S\left[t^nf^m(t)\right] = u^{n-m}\sum_{k=0}^{n-m} a_k^{n-m}u^kS_k\left[t^nf^m(t)\right]$$

where S_k denotes the k^{th} derivative of the function $t^nf^m(t)$ with respect to t.

Now, since we have developed the theorem, we need to make sure it's correct. We can perform proof by the method of induction, to make sure that it holds for n - m and also carries to n - m + 1.

Proof. Base case, we assume n=m for arbitrary m:

$$S\left[t^mf^m(t)\right] = u^{m-m}\sum_{k=0}^{m-m} a_k^{m-m}u^kS_k\left[t^mf^m(t)\right]$$

$$= u^0\sum_{k=0}^{0} a_k^0u^kS_k\left[t^mf^m(t)\right]$$

$$= u^0\sum_{k=0}^{0} a_k^0u^kS_k\left[t^1f^1(t)\right]$$

$$= u^0\left[a_0^0u^0S\left[t^mf^m(t)\right]\right] = S\left[t^mf^m(t)\right]$$

Induction Step.

Since it's true for $S\left[t^mf^m(t)\right]$ for n ≥ m for m, n ≥ 1, we'll now prove that it carries to $S\left[t^{n+1}f^m(t)\right]$ and also $S\left[t[t^nf^{m+1}(t)]\right]$ as well. Setting W(u) = $S\left[t^nf^m(t)\right]$ we have,

$$S\left[t^{n+1}f^m(t)\right] = S\left[t[t^nf^m(t)]\right] = u\frac{d(uW(u))}{du} = uW(u) + u^2\frac{d(W(u))}{du}$$

$$uW(u) + u^2W_1(u) = u^{n-m+1}\sum_{k=0}^{n-m} a_k^{n-m}u^kS_k\left[t^mf^m(t)\right] + u^2\frac{d}{du}\sum_{k=0}^{n-m} a_k^{n-m}u^{n+k}S_k\left[t^mf^m(t)\right]$$

$$= u^{n-m+1}\sum_{k=0}^{n-m} a_k^{n-m}u^kS_k\left[t^mf^m(t)\right] + u^2\sum_{k=0}^{n-m} a_k^{n-m}(n - m + k)u^{n+k+1}S_k\left[t^mf^m(t)\right] + u^{n+k}S_{k+1}\left[t^mf^m(t)\right]$$

$$= u^{n-m+1}\sum_{k=0}^{n-m} a_k^{n-m}u^kS_k\left[t^mf^m(t)\right] + u^{n-m+1}\sum_{k=0}^{n-m} a_k^{n-m}(n - m + k)u^kS_k\left[t^mf^m(t)\right] + u^{k+1}S_{k+1}\left[t^mf^m(t)\right]$$

$$= u^{n-m+1}\sum_{k=0}^{n-m}(n - m + k + 1)a_k^{n-m}u^kS_k\left[t^mf^m(t)\right] + u^{n-m+1}\sum_{k=0}^{n-m} a_k^{n-m}u^{k+1}S_k\left[t^mf^m(t)\right]$$

Noting that for k < 0, or k > n, $a_k^n = 0$, we can rewrite the previous equation as:

SJIF Impact Factor (2023): 8.574| ISI I.F. Value: 1.241| Journal DOI: 10.36713/epra2016 ISSN: 2455-7838(Online)
EPRA International Journal of Research and Development (IJRD)
Volume: 8 | Issue: 10 | October 2023 - Peer Reviewed Journal

$$= u^{n-m+1}\sum_{k=0}^{n-m+1}(n - m + k + 1)a_k^{n-m}u^kS_k\left[t^mf^m(t)\right] + u^{n-m+1}\sum_{k=0}^{n-m+1}a_{k+1}^{n-m}u^{k+1}S_k\left[t^mf^m(t)\right]$$

$$= u^{n-m+1}\sum_{k=0}^{n-m+1}\left[(n - m + k + 1)a_k^{n-m} + a_{k+1}^{n-m}\right]u^kS_k\left[t^mf^m(t)\right]$$

And since

$$a_k^{n-m} = a_{k-1}^{n-m-1} + (n - m + k)a_k^{n-m-1}$$

$$a_k^{n-m+1} = a_{k-1}^{n-m} + (n - m + k + 1)a_k^{n-m}$$

$$= u^{(n+1)-m}\sum_{k=0}^{(n+1)-m}a_k^{(n+1)-m}u^kS_k\left[t^mf^m(t)\right]$$

Case 2: We assume $S\left[t^mf^m(t)\right]$ to be true, now we'll prove it for $S\left[t^nf^{m+1}(t)\right]$.

$$= u^{n-(m+1)}\sum_{k=0}^{n-(m+!)}a_k^{n-(m+1)}u^kS_k\left[t^{m+1}f^{m+1}(t)\right]$$

8 Application of the Sumudu Transform to Differential Equations

Let us consider a general Euler equation. Euler equations are liner and homogenous ordinary differential equations of the form

$$t^2\frac{d^2y(t)}{dt^2} + \alpha t\frac{dy(t)}{dt} + \beta y(t) = R(t)$$

where α and β represent constants and R(t) represents a linear factor. Let us now consider a third-order differential equation:

$$t^3\frac{d^3y(t)}{dt^3} + ct^2\frac{d^2y(t)}{dt^2} + bt\frac{dy(t)}{dt} + ay(t) = R(t)$$

whose right-hand side, R(t), is a linear. If we apply Sumudu Transform on our equation it will remain unchanged since R(t) is a linear function. Let us examine an example.

$$2H(t) + 4tH'(t) + t^2H''(t) = 2$$

so if we apply the Sumudu Transform to the above second order linear differential equation, we get

$$2G(u) + 4uG'(U) + u^2G''(u) = 2$$

As we can see the equation remains unchanged generally. Hence there seems to be no direct advantage of using Sumudu transform in this case. On the other hand, much more mileage will be obtained if we recall the unit-preserving quality of Sumudu transform, especially in applications to the arguments of t and u, since they can be used interchangeably. So in cases where we can't find the Sumudu transform of an equation or there seems no advantage of using the Sumudu transform, one can look up for another equation and take its inverse Sumudu transform to solve. To illustrate the idea let us consider the example of a third-order Euler differential equation:

$$6H(t) + 18tH'(t) + 9t^2H''(t) + t^3H'''(t) = 6$$

Obviously, one solution of the obove equation is H(t) = 1. Now if we apply the Sumudu transform to the above equation, it will remain unchanged. So instead of applying the Sumudu transform we should find another way around it, which is to find a suitable format of an equation such that it can be readily inverted. Now if we take $S\left[t^nf(t)\right]$with n = 3 since it's a third-order differential equation, we get the equation whose coefficients matches exactly to our cooked-up example and setting $H(t) = S[h(s)]$, yields

$$S\left[s^3 h(s)\right] = t^3\left[6H(t) + 18tH_1(t) + 9t^2H_2(t) + t^3H_3(t)\right] = 6t^3 = S\left[s^3\right]$$

In this case, H(t) = h(s) = 1. Now let us consider another third order Euler-equation, but now with non-constant coefficients. This means that the coefficients will be different for the order of the equation.

$$18H'(u) + 18uH''(u) + 3u^2H'''(u) = 24$$

Again, if we apply the Sumudu transform to the above equation, it will remain unchanged since the right-hand side is a linear function. This brings us back to finding a suitable format for the equation so that it can be readily inverted. Now an interesting situation arises while we try to find a suitable inverting factor since the coefficients of the equation differ from the order of the equation. In cases like these, we can now rely on our developed theorem.

$$S\left[t^n f^m(t)\right] = u^{n-m}\sum_{k=0}^{n-m} a_k^{n-m} u^k S_k\left[t^n f^m(t)\right]$$

Now taking n=3 with m=1 and setting $H(w) = S[h'(s)]$, the above theorem yields

$$6H'(u) + 6uH''(u) + u^2H'''(u)$$

Now if we generalize this equation, we can observe that if we multiply the above equation with 3, it matches to the exact coefficients of our Euler equation. Thus we now have a suitable inverting factor. And setting $H(w) = S[h'(s)]$ we can find the transform of our equation by using the suitable inverting factor

$$S\left[s^3 h'(s)\right] = u^3\left[6H_1(u) + 6uH_2(u) + u^2H_3(u)\right] = 6w^3 = S\left[s^3\right]$$

Now, in this case, $H'(w) = h'(s) = 1$.

Now we can generalize that any ordinary linear different equation of the form

$$t^n \frac{d^m y(t)}{dt^m} + \frac{dy(t)}{dt} = 0$$

whose power of coefficients differ from the order of differential equation can now be solved by finding a suitable inverting factor.

The main point here is that unlike other transforms, the units-preservation property in combination with other properties of the Sumudu transform may allow us, according to the situation at hand, to transform the equation studied from the t domain to the u domain if the obtained equation is believed to be more accessible; or if necessary to consider the given equation as the Sumudu transform of another more readily solvable equation in the t domain, begotten by u inverse Sumudu transforming the equation at hand. So, the Sumudu transform may be used either way.

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